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Random Fibonacci sequences

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Abstract

Solutions to the random Fibonacci recurrence $x_{n+1} = x_n \pm \beta x_{n-1}$ decrease (increase) exponentially, $x_n \sim \exp(\lambda n)$, for sufficiently small (large) β . In the limits $\beta \rightarrow 0$ and $\beta \rightarrow \infty$, we expand the Lyapunov exponent $\lambda(\beta)$ in powers of β and β^{-1} , respectively. For the classical case of $\beta = 1$ we obtain exact non-perturbative results. In particular, an invariant measure associated with Riccati variable $r_n = x_{n+1}/x_n$ is shown to exhibit plateaux around all rational r .

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1. Introduction

The Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ... defined via $F_{n+1} = F_n + F_{n-1}$ abound in nature [1]³. For example, they govern the number of leaves, petals and seed grains in plants [2]; they also give the number of ancestors of a drone. The Fibonacci recurrence is the simplest recurrence in which each number depends on the previous two and this perhaps explains why the Fibonacci sequence is so ubiquitous. A natural *stochastic* modification of the Fibonacci sequence is to allow both additions and subtractions. Random Fibonacci sequences are related to many fields including condensed matter physics, dynamical systems, products of random matrices [3] and continued fractions. Random recurrences also form a chapter of the larger subject of iterated random functions [4].

The random Fibonacci recurrence $x_{n+1} = x_n \pm x_{n-1}$ results in sequences which behave erratically for small n . In the limit $n \rightarrow \infty$, however, exponential growth occurs with unit probability as was established by Furstenberg [5] in 1963. The large- n behaviour is characterized by the Lyapunov exponent λ ,

$$\lambda = \lim_{n \rightarrow \infty} \frac{\ln |x_n|}{n}. \quad (1)$$

³ <http://www.ee.surrey.ac.uk/Personal/R.Knott/> gives a huge amount of information on Fibonacci numbers. A mathematical periodical, the *Fibonacci Quarterly*, is entirely devoted to the subject.

Exponential growth (decay) means that λ is positive (negative). For the random Fibonacci recurrence where each \pm sign is independent and either $+$ or $-$ with probability $1/2$, the Lyapunov exponent is $\lambda = 0.123\,975\,59\dots$ [6]. Exponential growth may seem unsurprising but similar generalized random Fibonacci recurrence,

$$x_{n+1} = x_n \pm \beta x_{n-1} \quad (2)$$

gives exponential growth only when the parameter β is sufficiently large, $\beta > \beta_s \approx 0.702\,58$, whereas for $0 < \beta < \beta_s$ solutions decay exponentially [7]. The decay occurs even though the expected values $\langle x_n \rangle$ remain constant and the expected values of the higher-integer moments $\langle x_n^2 \rangle$, $\langle x_n^3 \rangle$ etc grow exponentially. Indeed, from equation (2) one can deduce $\langle x_{n+1}^2 \rangle = \langle x_n^2 \rangle + \beta^2 \langle x_{n-1}^2 \rangle$, which implies $\langle x_n^2 \rangle \sim \left[\frac{1}{2} + (\frac{1}{4} + \beta^2)^{1/2} \right]^n$, and similarly for higher moments [8]. Additionally, for $\beta > 1/4$ an interesting non-smooth dependence of the Lyapunov exponent on the parameter β has been observed [7], suggesting that the curve $\lambda(\beta)$ is a fractal (it remains unclear whether this curve becomes genuinely smooth for sufficiently large β). Similar non-smooth dependence on parameters has been reported for several disordered systems [9, 10]. Also, numerical results [7] suggest the following asymptotic behaviours of the Lyapunov exponent:

$$\lambda \approx \begin{cases} -\frac{1}{2} \beta^2 - \frac{15}{8} \beta^4 & \text{when } \beta \rightarrow 0 \\ \frac{1}{2} \ln \beta + \frac{0.114}{\beta} & \text{when } \beta \rightarrow \infty. \end{cases} \quad (3)$$

While understanding of the nature of the curve $\lambda(\beta)$ might be a very difficult problem, one should be able to carry out asymptotic expansions of the Lyapunov exponent using tools developed in studies of one-dimensional disordered systems [9–16]. Indeed, equation (2) admits the standard interpretation in terms of the one-dimensional (discretized) Schrödinger equation. A peculiarity of the present problem is that the corresponding random Hamiltonian is non-Hermitian. Non-Hermitian random Hamiltonians appear in various non-equilibrium problems [17] and exhibit interesting behaviours [18–21]; for example, a delocalization transition may occur even in one dimension. Fortunately, tools developed for Hermitian problems can often be extended to the non-Hermitian case.

In the next section, we employ perturbation theory to expand the Lyapunov exponent in powers of β . In particular, we show that in the $\beta \rightarrow 0$ limit the second term is $-1.75 \times \beta^4$ rather than $-1.875 \times \beta^4$ (cf equation (3)) and compute many more terms. In section 3, we use non-perturbative techniques to analyse the special $\beta = 1$ case. In section 4, we examine random sequences with Gaussian, rather than binary, disorder. Finally, a few open questions are discussed in the last section.

2. Perturbation theory

An old way of studying linear random recursions is to reduce them to random maps by introducing the Riccati variable $R_n = x_{n+1}/x_n$. In the present case, equation (2) becomes

$$R_n = 1 \pm \frac{\beta}{R_{n-1}}. \quad (4)$$

The Lyapunov exponent is given by

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \ln R_j. \quad (5)$$

In the large- n limit, the probability distribution of R_n becomes independent of n . The invariant distribution $P(R, \beta)$ satisfies

$$P(R) = \frac{\beta}{2(R-1)^2} \left[P\left(\frac{\beta}{R-1}\right) + P\left(\frac{\beta}{1-R}\right) \right] \tag{6}$$

which immediately follows from the random map (4). The analysis of the functional equation (6) often simplifies after transforming it into an integral equation

$$P(R, \beta) = \int dR' P(R', \beta) \frac{1}{2} \left[\delta\left(R - 1 + \frac{\beta}{R'}\right) + \delta\left(R - 1 - \frac{\beta}{R'}\right) \right]. \tag{7}$$

Once we know the invariant distribution $P(R, \beta)$, we can employ equation (5) to compute the Lyapunov exponent

$$\lambda(\beta) = \int dR P(R, \beta) \ln R. \tag{8}$$

Depending on the magnitude of β , the support of the invariant distribution $P(R, \beta)$ may be either finite or infinite. Let us assume the former. Then the extreme values satisfy $R_{\min} = 1 - \beta/R_{\min}$ and $R_{\max} = 1 + \beta/R_{\min}$, or

$$R_{\min} = \frac{1 + \sqrt{1 - 4\beta}}{2} \quad R_{\max} = \frac{3 - \sqrt{1 - 4\beta}}{2}. \tag{9}$$

Thus, the support is finite when the strength of disorder satisfies $\beta < 1/4$. Furthermore, in this case the support is not merely the interval $[R_{\min}, R_{\max}]$ but rather a fractal set similar to the Cantor set. Indeed, the map $R \rightarrow 1 \pm \beta/R$ transforms the interval $[R_{\min}, R_{\max}]$ into the union of two subintervals, $[R_{\min}, 1 - \beta/R_{\max}]$ and $[1 + \beta/R_{\max}, R_{\max}]$. Restricting the map to these two subintervals shows that they are transformed into four subintervals. Proceeding in the same manner *ad infinitum* we construct the fractal support of the invariant distribution. On the other hand, the support is unbounded when $\beta > 1/4$. This suggests employing different perturbation approaches for small and large β .

2.1. Weak-disorder expansion

When $\beta < 1/4$, it is desirable to compute $\lambda(\beta)$ without explicit determination of the invariant distribution $P(R, \beta)$ which is a complicated singular function. The trick is to transform the integral in the basic relation (8) into an integral which can be calculated perturbatively using only the normalization requirement $\int dR P(R) = 1$. To this end we insert equation (7) into (8) to yield

$$\lambda = \frac{1}{2} \int dR P(R) \left[\ln\left(1 - \frac{\beta}{R}\right) + \ln\left(1 + \frac{\beta}{R}\right) \right]. \tag{10}$$

Expanding the logarithms on the right-hand side of equation (10) we obtain

$$\lambda = -\frac{\beta^2}{2} \int dR R^{-2} P(R) - \frac{\beta^4}{4} \int dR R^{-4} P(R) + \dots$$

In the limit $\beta \rightarrow 0$, the interval $[R_{\min}, R_{\max}]$ shrinks to $R = 1$. Hence, the first integral on the right-hand side of the above equation approaches $\int dR P(R) = 1$. Thus $\lambda = -\beta^2/2 + \mathcal{O}(\beta^4)$ and the first term of the expansion was indeed derived without using an explicit form of the invariant distribution. However, the derivation of the next term is still impossible without knowledge of the invariant distribution. To avoid this we transform equation (10) as we transformed equation (8) into (10). Namely, we plug equation (7) into (10) to give

$$\lambda = \frac{1}{4} \int dR P(R, \beta) L_2(R, \beta) \tag{11}$$

where

$$L_2(R, \beta) = \ln \left(1 - \frac{\beta}{1 - \frac{\beta}{R}} \right) + \ln \left(1 - \frac{\beta}{1 + \frac{\beta}{R}} \right) + \ln \left(1 + \frac{\beta}{1 + \frac{\beta}{R}} \right) + \ln \left(1 + \frac{\beta}{1 - \frac{\beta}{R}} \right).$$

Expanding the logarithms and inserting this expansion into equation (11) we obtain

$$\lambda = -\frac{\beta^2}{2} \int dR P(R) - \frac{\beta^4}{4} \int dR \left(\frac{6}{R^2} + 1 \right) P(R) + \dots$$

The first integral is now identically equal to unity, while the second approaches seven in the small- β limit. Therefore, $\lambda = -\beta^2/2 - 7\beta^4/4 + \mathcal{O}(\beta^6)$ is the two-term expansion. This shows that the small- β expansion of equation (3) is slightly incorrect.

Repeating the above procedure k times, we recast equation (8) into

$$\lambda = 2^{-k} \int dR P(R, \beta) L_k(R, \beta) \tag{12}$$

where

$$L_k(R, \beta) = \sum_{\epsilon_1, \dots, \epsilon_k} \ln [1; \epsilon_1 \beta, 1, \epsilon_2 \beta, \dots, \epsilon_k \beta, R]. \tag{13}$$

The sum on the right-hand side of equation (13) is taken over all $\epsilon_j = \pm 1$, and $[1; a, b, \dots]$ denotes the continued fraction $1 + \frac{a}{b + \dots}$. Expanding $L_k(R, \beta)$ and plugging it into equation (12) one finds the correct expansion of the Lyapunov exponent up to $\mathcal{O}(\beta^{2k})$. A few of these terms can be computed by hand, and a little more can be done with the help of Mathematica. One obtains

$$\lambda(\beta) = -\frac{1}{2} \beta^2 - \frac{7}{4} \beta^4 - \frac{29}{3} \beta^6 - \frac{555}{8} \beta^8 - \frac{2843}{5} \beta^{10} - \frac{30755}{6} \beta^{12} + \mathcal{O}(\beta^{14}). \tag{14}$$

The radius of convergence of this series appears to be equal to one-quarter as one could guess from equations (9). Hence the Lyapunov exponent is an analytic function of β when $|\beta| < 1/4$. Amusingly, the invariant distribution is a very singular function in this range of the parameter β . The series (14) perfectly reproduces numerical results [7] (five representative digits), except for the case $\beta = 1/4$, for which $\lambda_{\text{num}} \approx -0.0429$ and $\lambda_{\text{theor}} \approx -0.0424$. This slight discrepancy is due to the fact that for $\beta = 1/4$ the generic term of the series decays as a power law in contrast with an exponential decay for $\beta < 1/4$.

2.2. Strong-disorder expansion

In the large- β limit, the support of the invariant distribution $P(R, \beta)$ is the whole real line. The trick which has been employed in the case of weak disorder does not apply; i.e., we cannot determine $\lambda(\beta)$ without knowledge of the invariant distribution. It proves convenient to use the modified Riccati variable $r_n = |x_{n+1}/x_n \sqrt{\beta}|$. Then, equation (2) reduces to the random map

$$r_n = \left| \frac{1}{r_{n-1}} \pm \delta \right| \quad \delta \equiv \beta^{-1/2}. \tag{15}$$

Once we know the invariant distribution $P(r, \delta)$, we can compute the Lyapunov exponent via equation (8), which now becomes

$$\lambda = \frac{1}{2} \ln \beta + \int dr P(r) \ln r. \tag{16}$$

The support of the invariant distribution is $0 \leq r < \infty$. It proves convenient to define $P(r)$ for negative r via $P(r) = P(-r)$, so the support is the entire real line. In present notations, the functional equation (6) becomes

$$2P(r) = \frac{1}{(r + \delta)^2} P\left(\frac{1}{r + \delta}\right) + \frac{1}{(r - \delta)^2} P\left(\frac{1}{r - \delta}\right). \tag{17}$$

For $\delta = 0$, this equation reduces to $P(r) = r^{-2}P(1/r)$, which has infinitely many solutions. For $\delta > 0$, however, the invariant distribution is unique. Thus taking the $\delta \rightarrow 0$ limit of the invariant distribution $P(r, \delta)$ we shall obtain a unique appropriate solution. In this sense, we are building a degenerate perturbation approach.

To determine $P(r, \delta)$, notice that equation (17) can be re-written as

$$P(r) = r^{-2}P(1/r) + \frac{1}{2} D_\delta^2[r^{-2}P(1/r)] \tag{18}$$

where D_δ^2 is the discrete analogue of the second derivative, $D_\delta^2 F(r) = F(r + \delta) - 2F(r) + F(r - \delta)$. Expanding the right-hand side of equation (18) into Taylor series and denoting $D = \frac{d}{dr}$ we obtain

$$P(r) = \sum_{k=0}^{\infty} \frac{\delta^{2k}}{(2k)!} D^{2k}[r^{-2}P(1/r)]. \tag{19}$$

The change of variable $r \rightarrow 1/r$ transforms equation (19) into

$$P(1/r) = \sum_{k=0}^{\infty} \frac{\delta^{2k}}{(2k)!} \mathcal{D}^{2k}[r^2 P(r)] \tag{20}$$

where $\mathcal{D} = r^2 \frac{d}{dr}$. If we now plug equation (20) into (19), we eliminate $P(1/r)$ and thus arrive at a closed equation for the invariant distribution $P(r, \delta)$:

$$P(r) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\delta^{2k+2l}}{(2k)!(2l)!} D^{2k}\{r^{-2}\mathcal{D}^{2l}[r^2 P(r)]\}. \tag{21}$$

Equation (21) suggests seeking a solution $P(r, \delta)$ which can be expanded as a power series in δ^2 :

$$P(r, \delta) = \sum_{j=0}^{\infty} \delta^{2j} P_j(r). \tag{22}$$

Plugging the series (22) into equation (21) one finds that the terms of unit order cancel. Equating terms of order δ^2 leads to the following second-order differential equation for $P_0(r)$:

$$D\mathcal{L}_1 P_0(r) = 0 \quad \text{with} \quad \mathcal{L}_1 \equiv D + r^2 D r^2. \tag{23}$$

Integrating once we arrive at $\mathcal{L}_1 P_0 = 0$, the integration constant being equal to zero to ensure that $P_0(r)$ vanishes in the large- r limit. This can be re-written as

$$(1 + r^4)P_0'(r) + 2r^3 P_0(r) = 0 \tag{24}$$

whose normalized solution is

$$P_0(r) = \frac{2\sqrt{\pi}}{\Gamma^2(1/4)} \frac{1}{\sqrt{1 + r^4}}. \tag{25}$$

Thus among infinitely many invariant distributions satisfying the duality relation $P(r) = r^{-2}P(1/r)$ we have indeed selected the specific solution (25). Note that $\int dr P_0(r) \ln r = 0$ (this is actually valid for any function which obeys the duality relation). Therefore, the anticipated contribution of unit order to the Lyapunov exponent (cf equation (16)) is actually equal to zero.

Similarly equating the terms of order δ^4 one finds $D\mathcal{L}_1 P_1 + D\mathcal{L}_3 P_0 = 0$, where \mathcal{L}_3 is a certain third-order differential operator. Integrating the above equation yields $\mathcal{L}_1 P_1 + \mathcal{L}_3 P_0 = 0$ (the integration constant is equal to zero to ensure that $P_1(r)$ vanishes in the large- r limit). Recalling that $\mathcal{L}_1 P_0 = 0$ one can simplify the term $\mathcal{L}_3 P_0$. The final first-order differential equation for $P_1(r)$ reads

$$\mathcal{L}_1 P_1(r) = \frac{(5 + r^4)D^3 + 2r^3 D^2}{12} P_0(r). \tag{26}$$

It is helpful to write $P_1(r) = P_0(r) f_1(r)$ and then exploit the identity $\mathcal{L}_1[P_0(r) f_1(r)] \equiv (1 + r^4)P_0(r) f_1'(r)$. Integrating the resulting equation yields

$$f_1(r) = c_1 + R_1(r) \quad R_1(r) = \frac{r^2(3r^8 + 8r^4 - 15)}{6(r^4 + 1)^3}. \tag{27}$$

The integration constant c_1 can be found from the relation $\int dr P_1(r) = 0$, which ensures that the normalization condition $\int dr P(r) = 1$ holds. We obtain

$$c_1 = \left[\frac{\Gamma(3/4)}{\Gamma(1/4)} \right]^2. \tag{28}$$

A direct computation gives $\int dr P_1(r) \ln r = c_1$. Thus, in this order the contribution to the Lyapunov exponent is equal to $c_1 \beta^{-1}$.

To determine $P_j(r)$, we repeat the above procedure. Plugging (22) into (21), equating terms of order δ^{2j+2} and integrating once the resulting equation yields

$$\mathcal{L}_1 P_j(r) + \sum_{n=1}^j \mathcal{L}_{2n+1} P_{j-n}(r) = 0 \tag{29}$$

where \mathcal{L}_{2n+1} is the differential operator of order $2n + 1$,

$$\mathcal{L}_{2n+1} = \frac{2}{(2n + 2)!} D^{2n+1} + \sum_{k=0}^n \frac{2}{(2k)!(2n - 2k + 2)!} D^{2k} \mathcal{D}^{2(n-k)+1} r^2.$$

Equations (29) can be solved recursively. Writing again $P_j(r) = f_j(r) P_0(r)$, we find in the second order

$$f_2(r) = -\frac{1}{96} + c_1 f_1(r) + R_2(r),$$

$$R_2(r) = \frac{r^4(463 - 3640r^4 + 2514r^8 + 440r^{12} - 17r^{16})}{24(r^4 + 1)^6}.$$

In the third order we have

$$f_3(r) = \frac{14\,699}{21\,600} c_1 - \frac{1}{96} f_1(r) + c_1 f_2(r) + R_3(r)$$

$$R_3(r) = \frac{r^2 Q_3(r)}{15\,120(r^4 + 1)^9}$$

$$Q_3(r) = 11\,340r^{32} - 678\,825r^{28} - 11\,260\,368r^{24} - 3619\,377r^{20} + 356\,871\,272r^{16} - 471\,736\,467r^{12} + 125\,696\,592r^8 - 5587\,155r^4 + 11\,340.$$

The constants were determined recursively after a long computation exploiting the normalization requirements $\int dr P_j(r) = 0$ and the basic identity $\Gamma(1 + z) = z\Gamma(z)$ for the gamma function [22].

After inserting the above expressions for $P_j(r)$ into equation (16), we arrive at the expansion

$$\lambda(\beta) = \frac{1}{2} \ln \beta + \sum_{k=1}^{\infty} a_k \beta^{-k}. \tag{30}$$

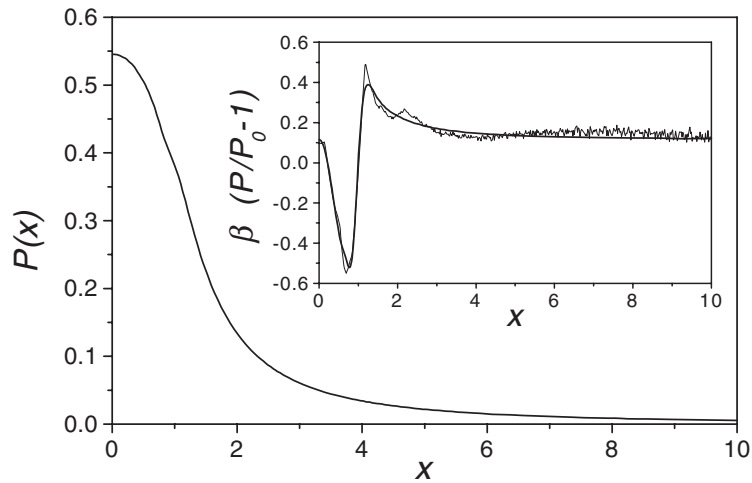


Figure 1. P_{num} , which is obtained after 2×10^9 iterations of the map (15), and P_{theor} , which is the exact third-order expansion in β , are plotted for $\beta = 10$ (curves are indistinguishable). The inset compares $\beta(P_{\text{num}}/P_0 - 1)$ to $\beta(P_{\text{theor}}/P_0 - 1)$ (thick curve). The Lyapunov exponent is $\lambda_{\text{num}} \approx \lambda_{\text{theor}} \approx 1.16293$.

One can in principle generate $\lambda(\beta)$ to any order. The first few coefficients are

$$\begin{aligned} a_1 &= c_1 = 0.114\,236\,645\,261\,1159\dots \\ a_2 &= c_1^2 + \frac{1}{144} = 0.019\,994\,455\,649\,58\dots \\ a_3 &= 0.010\,534\,5239\dots \\ a_4 &= 0.017\,663\,2096\dots \end{aligned}$$

The coefficients a_i have extremely complicated analytical expressions in terms of the Γ function, which we have not been able to simplify (although it appears plausible that these expressions can be reduced to polynomials of c_1 with rational coefficients). The exact value for a_1 is in good agreement with the numerical estimate from [7]. From the first four coefficients, one could guess that the radius of the convergence of series (30) is of order unity. However, an apparent fractal structure of the curve $\lambda(\beta)$ makes the above guess questionable. Even if our strong disorder expansion is asymptotic, it is quite accurate as can be seen by comparison of the four-term expansion (30) with numerical results. For $\beta = 8$, both give $\lambda \approx 1.05433$ whereas there is a slight discrepancy for $\beta = 4$ as $\lambda_{\text{num}} \approx 0.72309$ and $\lambda_{\text{theor}} \approx 0.72319$. In figure 1, we compare analytical (third-order expansion) and numerical results for the invariant distribution.

3. The golden mean case: $\beta = 1$

3.1. Generalities

We now focus on the particular case $\beta = 1$, which admits a non-perturbative treatment. An ingenious construction of the invariant distribution $P(r)$ and the invariant measure $\nu([a, b]) = \int_a^b dr P(r)$ which involves a Stern–Brocot division of the real line has been proposed by Viswanath [6]. In this section we first recall that construction [6] and the definition of the Stern–Brocot division (for details, see [23] and also [1], which describes closely related

Farey series). We then derive useful symmetry relations, which will lead to new quantitative results concerning the invariant distribution P and the invariant measure ν .

The invariant distribution $P(r)$ is symmetric, so we can limit ourselves to the half-line $r \geq 0$. The Stern–Brocot division is defined as follows: start with the half-line $I_0 = [\frac{0}{1}, \frac{1}{0}]$ and divide it into two intervals, $[\frac{0}{1}, \frac{1}{1}]$ and $[\frac{1}{1}, \frac{1}{0}]$; then divide every first-generation interval and continue in this manner. Generally, the mediant of an interval $[p/q, p'/q']$ is obtained by ‘Farey addition’:

$$\frac{p}{q} \oplus \frac{p'}{q'} = \frac{p+p'}{q+q'}. \tag{31}$$

Thus, the interval $[p/q, p'/q']$ is divided according to the rule

$$\left[\frac{p}{q}, \frac{p'}{q'} \right] \longrightarrow \left[\frac{p}{q}, \frac{p+p'}{q+q'} \right] \cup \left[\frac{p+p'}{q+q'}, \frac{p'}{q'} \right]. \tag{32}$$

This generates 2^n intervals at the n th generation, each interval I of the $(n - 1)$ th generation producing a left and a right child ($\mathcal{L}I$ and $\mathcal{R}I$) in the n th generation, according to equation (32). For instance, $[\frac{1}{3}, \frac{1}{2}] = \mathcal{R}\mathcal{L}\mathcal{L}I_0$, and generally every Stern–Brocot interval I can be presented as $I = \mathcal{S}I_0$, where \mathcal{S} is the unique sequence of \mathcal{L} and \mathcal{R} . The length of \mathcal{S} is denoted $|\mathcal{S}|$. Using this representation one can prove numerous properties of the Stern–Brocot division, for example the assertion that for every Stern–Brocot interval $[\frac{p}{q}, \frac{p'}{q'}]$, the identity $p'q - pq' = 1$ holds [23].

The invariance condition, i.e. equation (17) with $\delta = 1$, can be re-written in terms of the invariant measure to give

$$\nu(a, b) = \frac{1}{2} \left\{ \nu \left(\frac{1}{-1+b}, \frac{1}{-1+a} \right) + \nu \left(\frac{1}{1+b}, \frac{1}{1+a} \right) \right\}.$$

Making use of the above equation, it is possible to express the measure of the left and right children via the measure of the parent interval [6]:

$$\nu(\mathcal{L}\mathcal{S}I_0) = \begin{cases} (1 - \tau) \nu(\mathcal{S}I_0) & \text{if } |\mathcal{S}| \text{ is even} \\ \tau \nu(\mathcal{S}I_0) & \text{if } |\mathcal{S}| \text{ is odd} \end{cases} \tag{33}$$

and

$$\nu(\mathcal{R}\mathcal{S}I_0) = \begin{cases} \tau \nu(\mathcal{S}I_0) & \text{if } |\mathcal{S}| \text{ is even} \\ (1 - \tau) \nu(\mathcal{S}I_0) & \text{if } |\mathcal{S}| \text{ is odd} \end{cases} \tag{34}$$

where $\tau = (\sqrt{5} - 1)/2$ is the golden ratio (which is also the inverse growth constant of the deterministic Fibonacci numbers, $F_n \sim \tau^{-n}$).

We now turn to the derivation of symmetry relations, which will be helpful later. To obtain the first one we take an arbitrary Stern–Brocot interval $\mathcal{S}I_0 = [\frac{p}{q}, \frac{p'}{q'}]$ and notice that the Stern–Brocot interval $\mathcal{S}\mathcal{R}\mathcal{R}I_0 = [\frac{p}{q} + 2, \frac{p'}{q'} + 2]$ differs from $\mathcal{S}I_0$ by a mere translation. Using equations (33), (34) one finds $\tau(1 - \tau) \nu(\mathcal{S}I_0) = \nu(\mathcal{S}\mathcal{R}\mathcal{R}I_0)$, i.e.

$$\nu \left(\left[\frac{p}{q} + 2, \frac{p'}{q'} + 2 \right] \right) = \rho \nu \left(\left[\frac{p}{q}, \frac{p'}{q'} \right] \right) \tag{35}$$

with $\rho = \tau(1 - \tau) = \sqrt{5} - 2$.

Note that every interval with rational end points can be formed by joining appropriate Stern–Brocot intervals. Hence equation (35) holds for any rational interval $[\frac{p}{q}, \frac{p'}{q'}]$. Since rational numbers form a dense set on the line and $\nu(r) \equiv \nu([0, r])$ is a continuous (increasing)

function, equation (35) applies to arbitrary intervals. Specializing to the interval $[0, r]$, we arrive at the first relation

$$v([2, r + 2]) = \rho v(r) \tag{36}$$

which can be also presented as

$$P(r + 2) = \rho P(r) \quad \text{for } r \geq 0. \tag{37}$$

The second symmetry relation,

$$P(2 - r) = P(r) \quad \text{for } 0 \leq r \leq 2 \tag{38}$$

expresses a mirror symmetry with respect to unity in the interval $[0, 2]$. To prove this result, we take a Stern–Brocot interval $\mathcal{SL}I_0 = [p/q, p'/q'] \subset [0, 1]$ and notice that its symmetric image $[2 - p'/q', 2 - p/q] \subset [1, 2]$ is also a Stern–Brocot interval. Specifically, the symmetric interval can be presented as $\bar{\mathcal{S}}\mathcal{LR}I_0$, where $\bar{\mathcal{S}}$ is the sequence obtained from \mathcal{S} by exchanging the \mathcal{L} and \mathcal{R} . It turns out that the interval and its symmetric image have the same invariant measures

$$v(\mathcal{SL}I_0) = v(\bar{\mathcal{S}}\mathcal{LR}I_0). \tag{39}$$

One then deduces equation (38) from (39), repeating the argument which has been employed in deducing equation (37) from (35).

The proof of equation (39) can be accomplished by induction on the length. Suppose that equation (39) holds for sequences \mathcal{S} of length $< n$. Assuming n even and taking the right-hand child of $\mathcal{SL}I_0$, we use equation (34) to yield

$$v(\mathcal{RSL}I_0) = (1 - \tau) v(\mathcal{SL}I_0) \tag{40}$$

$$v(\mathcal{L}\bar{\mathcal{S}}\mathcal{LR}I_0) = (1 - \tau) v(\bar{\mathcal{S}}\mathcal{LR}I_0). \tag{41}$$

Since $\mathcal{L} = \bar{\mathcal{R}}$ and the right-hand sides of the above equations are equal, we obtain $v(\mathcal{RSL}I_0) = v(\bar{\mathcal{R}}\bar{\mathcal{S}}\mathcal{LR}I_0)$, thus proving the induction step for even n and the right child. The three other cases can be proved in a similar way.

3.2. Analytical results

In this section, we shall make use of the above symmetries to obtain quantitative results for the invariant distribution P and the invariant measure v . For instance, one can readily establish extreme behaviours. From equations (33) and (34), one finds

$$v(\mathcal{L}^{2n}I_0) = v(\mathcal{R}^{2n}I_0) = \rho^n v(I_0) \tag{42}$$

for any (positive) integer n . Clearly, $\mathcal{L}^{2n}I_0 = [0, \frac{1}{2n}]$ and $\mathcal{R}^{2n}I_0 = [2n, \infty]$. Additionally, $v(I_0) = 1/2$, which immediately follows from normalization. Thus equation (42) reduces to

$$v\left(\frac{1}{2n}\right) = v([2n, \infty]) = \frac{1}{2} \rho^n. \tag{43}$$

From equation (43) we deduce the small- r behaviour,

$$v(r) \sim e^{-c/r} \quad \text{with } c = -\frac{1}{2} \ln \rho \tag{44}$$

and the large- r behaviour,

$$\frac{1}{2} - v(r) \sim e^{-cr}. \tag{45}$$

Thus, the invariant measure $v(r)$ sharply vanishes when $r \rightarrow 0$. Remarkably, similar plateaux exist around all rational r . It is useful to demonstrate the existence of the plateau in the proximity of a simple rational point, say $r = 1/2$. Take the family of intervals

$\mathcal{L}^{2n} \mathcal{R} \mathcal{L} I_0 = [\frac{1}{2}, \frac{2n+1}{4n+1}]$. The length of the n th interval is $\epsilon_n = [2(4n + 1)]^{-1}$. Exploiting equations (33) and (34), we can compute the measure of these intervals and then examine how the measure scales with length. We obtain

$$v \left(\left[\frac{1}{2}, \frac{2n+1}{4n+1} \right] \right) = \frac{(1-\tau)^2}{2} \rho^n \sim e^{-c/4\epsilon_n}. \tag{46}$$

The continuity of the measure implies that the scaling holds for all small intervals, $v(\frac{1}{2} + \epsilon) - v(\frac{1}{2}) \sim e^{-c/4\epsilon}$.

Similarly, one can work out the proximity of an arbitrary rational number. One finds that the measure of the interval $[p/q, p/q + \epsilon]$ (p and q are mutually prime integers) exhibits the following behaviour:

$$v \left(\frac{p}{q} + \epsilon \right) - v \left(\frac{p}{q} \right) \sim \exp \left(-\frac{c}{q^2 \epsilon} \right) \tag{47}$$

with $c = -\frac{1}{2} \ln \rho$ as above. Similar asymptotics apply when we approach the rational point from the left.

Turn now to the invariant distribution $P(r)$. This function obeys a striking number of intricate identities. (Note that it is a slight abuse of language to speak of P as a function: P is a distribution rather than a function.) In the following, we shall be using equation (17), with $\delta = 1$, so we write it down for easy reference:

$$2P(r+1) = \frac{1}{r^2} P \left(\frac{1}{r} \right) + \frac{1}{(r+2)^2} P \left(\frac{1}{r+2} \right). \tag{48}$$

Note that equation (48) at $r + 2n$ can be written as

$$2P(r+2n+1) = U_n(r) + U_{n+1}(r) \tag{49}$$

where we used the shorthand notation

$$U_n(r) = \frac{1}{(r+2n)^2} P \left(\frac{1}{r+2n} \right). \tag{50}$$

Changing n to $n + 1$ in equation (49) gives

$$2P(r+2n+3) = U_{n+1}(r) + U_{n+2}(r). \tag{51}$$

Recalling that $P(r+2n+3) - \rho P(r+2n+1) = 0$ (this is equation (37) at $r + 2n$), we reduce (49), (51) to a recursion which involves only U :

$$U_{n+2}(r) + (1-\rho)U_{n+1}(r) - \rho U_n(r) = 0. \tag{52}$$

Since the variable r is mute, equation (52) is merely a linear recursion, which is straightforwardly solved to find two independent solutions, $(-1)^n$ and ρ^n . Therefore, the general solution is

$$\frac{1}{(r+2n)^2} P \left(\frac{1}{r+2n} \right) = A(r)(-1)^n + B(r)\rho^n. \tag{53}$$

In the $n \rightarrow \infty$ limit, the left-hand side of equation (53) approaches zero. Thus, $A(r) = 0$. The other coefficient $B(r)$ is found by specializing equation (53) to $n = 0$. Equation (53) therefore reduces to

$$\frac{1}{(r+2n)^2} P \left(\frac{1}{r+2n} \right) = \frac{\rho^n}{r^2} P \left(\frac{1}{r} \right). \tag{54}$$

One can derive a few more useful formulae relating $P(r)$ at different points. Performing the change of variable $r \rightarrow r^{-1}$, one transforms equation (54) into

$$P(r) = \frac{\rho^{-n}}{(1+2nr)^2} P \left(\frac{r}{1+2nr} \right). \tag{55}$$

One can also take equation (54) at $n = 1$ and insert it into equation (48). The outcome reads

$$P(r + 1) = \frac{1 + \rho}{2r^2} P\left(\frac{1}{r}\right). \tag{56}$$

Take now equation (56), change the variable $r \rightarrow r + 1$ and use equation (37). This transforms equation (56) into

$$P(r) = \left(\frac{1 + \tau}{1 + r}\right)^2 P\left(\frac{1}{1 + r}\right). \tag{57}$$

We can take the same identity with $(1 + r)^{-1}$ instead of r and insert it into the right-hand side of equation (57), thus obtaining another identity. Proceeding in this manner, we arrive at a series of identities

$$P(r) = \frac{(1 + \tau)^{2m}}{(F_m + r F_{m-1})^2} P\left(\frac{F_{m-1} + r F_{m-2}}{F_m + r F_{m-1}}\right). \tag{58}$$

These identities apply for all integer m as well as the Fibonacci numbers F_m which are defined for all integer m , for example $F_{-4} = 2, F_{-3} = -1, F_{-2} = 1, F_{-1} = 0, F_0 = 1$.

Replacing $r \rightarrow 2 - r$ in identities (58) and using the symmetry relation (38) one obtains another infinite series of identities. The simplest such identity (corresponding to $m = 1$) reads

$$P(r) = \left(\frac{1 + \tau}{3 - r}\right)^2 P\left(\frac{1}{3 - r}\right). \tag{59}$$

The above identities together with the basic relation $\nu([a, b]) = \int_a^b dr P(r)$ can be used to re-derive all previous results about the invariant measure, including the most interesting findings about the plateaux.

3.3. Generating the invariant measure

Because of symmetries, it is sufficient to study $\nu(r)$ on the interval $[0, 1]$. To distinguish the restricted version from the general case we change the notation: $r \rightarrow x, P \rightarrow \frac{1-\rho}{4} f, \nu \rightarrow \frac{1-\rho}{4} N$. The prefactor $(1 - \rho)/4$ guarantees that the invariant measure $N(x) = \int_0^x dy f(y)$ obeys $N(1) = 1$. To see this, we exploit the normalization condition, $1 = \int_{-\infty}^{\infty} dr P(r)$:

$$2 \int_0^{\infty} dr P(r) = \frac{2}{1 - \rho} \int_0^2 dr P(r) = \frac{4}{1 - \rho} \int_0^1 dr P(r).$$

The first identity above is obtained by using equation (37) and summing up the geometric series while the last is an obvious consequence of equation (38). Thus we indeed obtain $N(1) = 1$ if we choose $f(x) = \frac{4}{1-\rho} P(x)$.

Now let us integrate $f(x)F(x)$ with any $F(x)$ over intervals $(0, 1/3), (1/3, 1/2), (1/2, 1)$. We find

$$\begin{aligned} \int_0^{1/3} dx f F &= \int_0^1 \frac{dx}{(1 + 2x)^2} f\left(\frac{x}{1 + 2x}\right) F\left(\frac{x}{1 + 2x}\right) \\ &= \rho \int_0^1 dx f(x) F\left(\frac{x}{1 + 2x}\right) \end{aligned} \tag{60}$$

$$\begin{aligned} \int_{1/3}^{1/2} dx f F &= \int_0^1 \frac{dx}{(3 - x)^2} f\left(\frac{1}{3 - x}\right) F\left(\frac{1}{3 - x}\right) \\ &= (1 + \tau)^{-2} \int_0^1 dx f(x) F\left(\frac{1}{3 - x}\right) \end{aligned} \tag{61}$$

$$\begin{aligned} \int_{1/2}^1 dx f F &= \int_0^1 \frac{dx}{(1+x)^2} f\left(\frac{1}{1+x}\right) F\left(\frac{1}{1+x}\right) \\ &= (1+\tau)^{-2} \int_0^1 dx f(x) F\left(\frac{1}{1+x}\right). \end{aligned} \quad (62)$$

In deriving the first lines in the above formulae we have used the mappings $x(1+2x)^{-1}$, $(3-x)^{-1}$, $(1+x)^{-1}$ of the unit interval $(0, 1)$ onto the intervals which appear on the left-hand side. We then exploited equation (55) at $n = 1$, equation (59) and equation (57), respectively, to obtain the second lines.

Summing up equations (60)–(62) leads to the identity

$$\int_0^1 dx f(x) F(x) = \int_0^1 dx f(x) \hat{T}[F(x)] \quad (63)$$

where the linear operator \hat{T} acts on F according to

$$\hat{T}[F(x)] = \sum_{i=1}^3 \alpha_i F(h_i(x)) \quad (64)$$

$$\alpha_1 = \rho \quad \alpha_2 = \alpha_3 = (1+\tau)^{-2} \quad \sum_{i=1}^3 \alpha_i = 1 \quad (65)$$

$$h_1(x) = \frac{x}{1+2x} \quad h_2(x) = \frac{1}{3-x} \quad h_3(x) = \frac{1}{1+x}. \quad (66)$$

These relations show that the distribution f can be generated by the following simple recursion process. We take an arbitrary point $x \in (0, 1]$ and assign a unit weight $w(x) = 1$. We then define three new points and associated weights according to the rule

$$x_i = h_i(x) \quad w(x_i) = \alpha_i w(x) \quad i = 1, 2, 3. \quad (67)$$

Iterating n times, we obtain 3^n different x_i . The weights $w(x_i)$ of these points add up to unity and $w(x)$ converge (in the weak sense) to $f(x)$. In figure 2, we plot the invariant measure obtained after four and ten iterations.

The plateaux described by equation (47) are clearly visible in figure 2. The smaller the denominator q , the wider is the associated plateau as predicted by equation (47). In a pseudo-gap of f (associated with a plateau for N), $N(p/q) = \{n_{p/q}\tau\}$, where $\{\cdot\}$ denotes the positive fractional part and $n_{p/q}$ is a possibly negative integer. The values of $N(p/q)$ for a few small denominator fractions are presented in figure 2; they have been calculated by using equations (63)–(66). Thus, a kind of ‘pseudo-gap labelling theorem’ [24] applies here: for tight-binding Hamiltonians on 1D quasiperiodic chains associated with an irrational number α (see [24] for a general formulation), this theorem states that the integrated density of states (IDOS) displays plateaux (corresponding to the energy true gaps). In these plateaux, the IDOS takes values precisely of the form $\{n\alpha\}$. In one dimension, the IDOS is intimately related to the invariant measure ν [16]. Therefore, it is not surprising that ν exhibits a similar behaviour.

We now show how to calculate the Lyapunov exponent from the restricted invariant distribution $f(x)$. Take the basic formula, $\lambda = \int dr P(r) \ln r$, and perform the same operations as we did for the normalization condition at the beginning of this section. We obtain

$$\begin{aligned} \lambda &= \int_0^1 dx f(x) g(x) + \mu \\ g(x) &= \frac{1-\rho}{2} \sum_{k=0}^{\infty} \rho^k \ln \left(1 - \left(\frac{1-x}{2k+1} \right)^2 \right) \end{aligned}$$

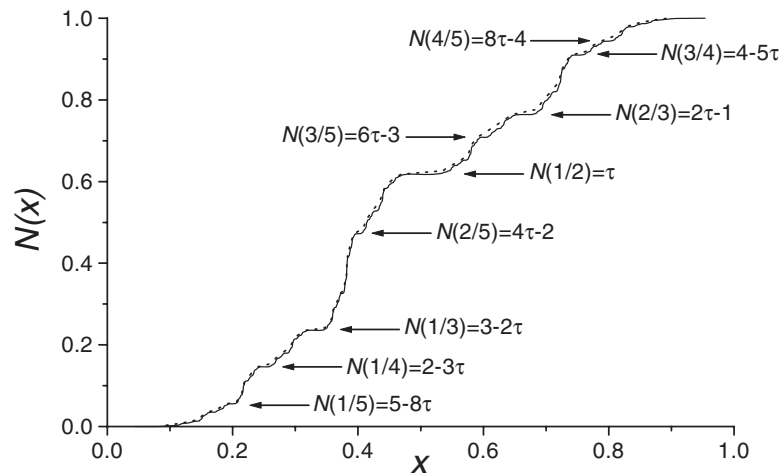


Figure 2. The invariant measure $N(x) = \int_0^x dy f(y)$ obtained after four iterations (dotted curve involving only $3^4 = 81$ different x_i) and ten iterations ($3^{10} = 59\,049$ different x_i).

$$\mu = (1 - \rho) \sum_{k=0}^{\infty} \rho^k \ln(2k + 1) = 0.293\,204\,911\,37\dots$$

We then numerically compute λ via

$$\lambda = \lim_{n \rightarrow \infty} \sum_{i=1}^{3^n} w(x_i) g(x_i) + \mu.$$

For instance, starting from $x = 1 - \tau$, one reproduces ten digits of λ after only eight iterations.

4. Gaussian random sequences

Invariant measures exhibited by the Fibonacci random sequences appear fractal for all strengths of disorder. This is apparently caused by the discreteness of the disorder. Additionally, there is a transition between weak disorder, manifested by an invariant measure whose support is bounded, and strong disorder, characterized by an invariant measure with unbounded support. This feature is seemingly caused by the boundness of the disorder. Hence, random sequences with unbounded continuous disorder distributions provide a natural counterpart to random sequences with binary disorder. As a specific example, we consider the Gaussian random recurrence,

$$x_{n+1} = x_n + \beta z_n x_{n-1} \tag{68}$$

where z_n are independent identically distributed random variables with Gaussian probability density

$$G(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}. \tag{69}$$

To analyse the Gaussian random recurrence, we again use the Riccati variable $R_n = x_{n+1}/x_n$, which reduces equation (68) to the random map

$$R_n = 1 + \frac{\beta z_n}{R_{n-1}}. \tag{70}$$

The invariant distribution $P(R)$ satisfies

$$P(R) = \int dz G(z) \int dR' P(R') \delta \left(R - 1 - \frac{\beta z}{R'} \right) \tag{71}$$

which can be re-written as

$$P(R + 1) = \frac{1}{\sqrt{2\pi\beta^2}} \int dR' |R'| P(R') e^{-R'^2 R^2 / 2\beta^2}. \tag{72}$$

Note two general features of the invariant distribution which directly follow from equation (72). One is a power-law large- R asymptotics,

$$P(R) \approx \sqrt{\frac{2}{\pi}} \frac{\beta P(0)}{R^2} \quad \text{for } R \gg \beta. \tag{73}$$

Another unexpected feature is a weak logarithmic divergence at $R = 1$:

$$P(R) \approx \frac{2}{\pi} P(0) \ln \left(\frac{1}{|R - 1|} \right) \quad \text{for } |R - 1| \ll \beta. \tag{74}$$

Equation (74) follows from equations (72) and (73).

We now turn to a perturbative analysis. While for the Gaussian random recurrence there appears to be no threshold separating weak and strong disorder, different approaches should be implemented when $\beta \rightarrow 0$ and $\beta \rightarrow \infty$, respectively. In the former region, the regular perturbation theory applies while in the latter region one needs a singular perturbation theory.

4.1. Weak disorder

Equation (70) shows that for weak disorder ($\beta \rightarrow 0$) the magnitude of $R - 1$ is comparable to β . This suggests rescaling the variable R and the invariant distribution $P(R)$ according to

$$R = 1 + \beta r \quad P(R) = \beta^{-1} Q(r). \tag{75}$$

The normalization condition $\int dR P(R) = 1$ now reads

$$\int dr Q(r) = 1. \tag{76}$$

The governing equation (72) becomes

$$\frac{Q(r)}{G(r)} = \int_{-\infty}^{\infty} dr' |1 + \beta r'| Q(r') e^{-r^2(\beta r' + \beta^2 r'^2/2)}. \tag{77}$$

In the small- β limit, we simplify equation (77) to

$$\frac{Q(r)}{G(r)} = \int_{-\infty}^{\infty} dr' (1 + \beta r') Q(r') e^{-r^2(\beta r' + \beta^2 r'^2/2)}. \tag{78}$$

The error caused by the above simplification is of order $Q(-1/\beta)$. We shall see that it vanishes as $\exp(-1/2\beta^2)$ and therefore it can be ignored in perturbative analysis.

We are seeking a perturbative solution. The symmetry $\beta \leftrightarrow -\beta$ suggests an expansion in β^2 rather than β :

$$Q(r) = \sum_{n=0}^{\infty} \beta^{2n} Q_n(r). \tag{79}$$

We must also expand the exponent on the right-hand side of equation (77). This exponent is the generating function of the Hermite polynomials $H_n(r)$,

$$\exp[-r^2(x + x^2/2)] = \sum_{n=0}^{\infty} \frac{(-xr)^n}{n!} H_n(r) \tag{80}$$

where $H_n(r)$ are defined via [22]

$$H_n(r) = e^{r^2/2} \left(-\frac{d}{dr}\right)^n e^{-r^2/2}. \tag{81}$$

Thus, the Hermite polynomials make their appearance in the problem with non-Hermitian Hamiltonian.

Inserting equations (79) and (80) into (78), we obtain

$$\frac{Q_n(r)}{G(r)} = \sum_{k=1}^n \frac{r^{2k-1}}{(2k)!} H_{2k+1}(r) \int dr' (r')^{2k} Q_{n-k}(r'). \tag{82}$$

In deriving equation (82), we also used the recursion relation for the Hermite polynomials [22]:

$$H_{n+1}(r) = r H_n(r) - n H_{n-1}(r). \tag{83}$$

Solving equation (82) recursively yields

$$Q_0(r) = G(r)$$

$$\frac{Q_1(r)}{G(r)} = \frac{1}{2} r H_3(r)$$

$$\frac{Q_2(r)}{G(r)} = \frac{3}{2} r H_3(r) + \frac{1}{8} r^3 H_5(r)$$

$$\frac{Q_3(r)}{G(r)} = 12 r H_3(r) + \frac{5}{4} r^3 H_5(r) + \frac{1}{48} r^5 H_7(r)$$

$$\frac{Q_4(r)}{G(r)} = \frac{7857}{32} r H_3(r) + \frac{135}{8} r^3 H_5(r) + \frac{7}{16} r^5 H_7(r) + \frac{1}{384} r^7 H_9(r)$$

$$\frac{Q_5(r)}{G(r)} = \frac{1362\,843}{256} r H_3(r) + \frac{48\,675}{128} r^3 H_5(r) + \frac{147}{16} r^5 H_7(r) + \frac{3}{32} r^7 H_9(r) + \frac{1}{3840} r^9 H_{11}(r)$$

etc. We shall use these results to compute the Lyapunov exponent. The basic formula (8) now reads

$$\lambda = \int_{-\infty}^{\infty} dr Q(r) \ln |1 + \beta r|. \tag{84}$$

Expanding the logarithm and the invariant distribution, equation (79), and recalling that $Q(r) = Q(-r)$, we obtain

$$\lambda = - \sum_{n=1}^{\infty} \beta^{2n} \sum_{k=1}^n \frac{1}{k} \int_0^{\infty} dr r^{2k} Q_{n-k}(r). \tag{85}$$

Inserting the above expressions for Q_n ($n = 0, \dots, 5$) into equation (85), we obtain the weak-disorder expansion:

$$\lambda(\beta) = -\frac{1}{2} \beta^2 - \frac{9}{4} \beta^4 - 22 \beta^6 - \frac{13\,197}{32} \beta^8 - \frac{2374\,335}{256} \beta^{10} - \frac{118\,392\,093}{512} \beta^{12} + \mathcal{O}(\beta^{14}). \tag{86}$$

Interestingly, neither the R^{-2} asymptotics (73) nor the logarithmic singularity (74) appear in the weak-disorder expansion. Both these behaviours are non-perturbative. For instance, $P(0)$, which appears on the right-hand sides of equations (73) and (74), scales as $\exp(-1/2\beta^2)$; i.e., the behaviours (73) and (74) are beyond the scope of perturbation techniques. Note also that the observation of the logarithmic singularity (74) requires probing of a prohibitively tiny region $r \sim \exp[-\exp(-1/2\beta^2)]$.

The occurrence of non-perturbative corrections (of order $\exp(-1/2\beta^2)$) suggests that the radius of convergence of the series (86) is equal to zero. This is (non-rigorously) confirmed by the following approximate analysis of $\lambda(\beta)$. First, we write $R_n \approx 1 + \beta z_n$, as the distribution

of R_{n-1} is strongly peaked at $R = 1$ for small β . Within this approximation, and using the fact that the Gaussian distribution is an even function, we obtain

$$\lambda(\beta) \approx \frac{1}{2} \int_{-\infty}^{\infty} dz G(z) \ln |1 - \beta^2 z^2| dz \tag{87}$$

which reproduces exactly the first term of equation (86). Now, expanding equation (87) in powers of β^2 is likely to lead to the correct qualitative behaviour for the full general expansion. The generic term,

$$-\frac{\beta^{2n}}{2n} \int_{-\infty}^{\infty} dz G(z) z^{2n} = -\frac{\beta^{2n} 2^{n-1}}{n \sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right)$$

grows faster than any exponential, ensuring that the radius of convergence is indeed zero. Of course, we cannot exclude that for the actual expansion, subtle cancellations lead to a finite radius of convergence. However, the occurrence of non-perturbative corrections, and concrete ingredients of the argument presented above (mainly, the unboundness of the distribution $G(z)$ and the fact that the series $\ln(1 + x)$ has a finite radius of convergence) which seem to persist in the general case, favour a zero radius of convergence and the asymptotic character of the series (86). This is very different from the case of random Fibonacci sequences, where the weak-disorder expansion has a finite radius of convergence and there was no trace of any non-perturbative contribution.

4.2. Strong disorder

For $\beta \rightarrow \infty$, we again use the properly normalized Riccati variable $y_n = x_{n+1}/x_n \sqrt{\beta}$. Equation (68) reduces to the random map

$$y_n = \frac{z_n}{y_{n-1}} + \delta \quad \delta \equiv \beta^{-1/2}. \tag{88}$$

The invariant distribution satisfies

$$P(y + \delta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\eta |\eta| P(\eta) \exp\left\{-\frac{y^2 \eta^2}{2}\right\} \tag{89}$$

and the Lyapunov exponent is given by equation (16) as in the Fibonacci case. Equation (89) suggests seeking a perturbative solution. In the zeroth order, one might set $\delta = 0$ in equation (89). The corresponding invariant distribution $P_0(y)$ is an even function of y which satisfies

$$P_0(y) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} d\eta \eta P_0(\eta) \exp\left\{-\frac{y^2 \eta^2}{2}\right\}. \tag{90}$$

Paradoxically, a (formal) solution to this equation,

$$P_0(y) = \frac{A}{|y|} \tag{91}$$

does not obey the normalization requirement. This indicates that the naive perturbation approach does not work and one must develop a singular perturbation theory. One still anticipates that $P_0(y)$ is given by equation (91) apart from the small and large scales, $|y| \sim \delta$ and $|y| \sim \delta^{-1}$, which are implied by the random map (88). Treating these scales as cutoffs allows us to normalize the solution (91) and to estimate the amplitude $A \approx (2 \ln \beta)^{-1}$. One can establish the existence of the cutoffs more rigorously. Using equations (89) and (91) one estimates $P(0) \sim A\delta^{-1}$, in agreement with the existence of the small-scale cutoff $y \sim \delta$. The large-scale cutoff already follows from equation (73), which now reads

$$P(y) \rightarrow \sqrt{\frac{2}{\pi}} \frac{P(0)}{y^2} \quad \text{for } y \gg \delta^{-1}. \tag{92}$$

Note also that the deficiency of the naive perturbation approach is clear from the respective random map, $y_n = z_n/y_{n-1}$. Indeed, iterating the above map and taking the logarithm gives $\ln y_n = \sum (-1)^{n-k} \ln z_k$. The central limit theorem now asserts that the scale of the limiting distribution grows indefinitely with n , namely $\ln y_n \sim \sqrt{n}$. Thus for $\delta = 0$ already the basic assumption that y_n approaches a limiting distribution which does not depend on n is incorrect.

We now present a computation of the zeroth-order contribution to the Lyapunov exponent which does not require the knowledge of P_0 . Denote by Λ the zeroth-order contribution to the Lyapunov exponent. We have

$$\begin{aligned} \Lambda &= 2 \int_0^\infty dy P_0(y) \ln y \\ &= 2\sqrt{\frac{2}{\pi}} \int_0^\infty d\eta \eta P_0(\eta) \int_0^\infty dy \ln y \exp\left\{-\frac{y^2\eta^2}{2}\right\} \\ &= 2\sqrt{\frac{2}{\pi}} \int_0^\infty d\eta P_0(\eta) \int_0^\infty \frac{dt}{\sqrt{2t}} e^{-t} \ln\left(\frac{\sqrt{2t}}{\eta}\right) \\ &= \int_0^\infty d\eta P_0(\eta) [\Psi(\frac{1}{2}) + \ln 2 - 2 \ln \eta] \\ &= \frac{1}{2} [\Psi(\frac{1}{2}) + \ln 2] - \Lambda. \end{aligned}$$

In the second line we used equation (90); this step is not really rigorous though we think the final result is correct. The variable t which appears in the third line has been defined via $t = y^2\eta^2/2$; in the fourth line we used the digamma (psi) function, $\Psi(x) = \Gamma'(x)/\Gamma(x)$ [22]; in the last line we used the normalization requirement and the definition of Λ . The above equation yields Λ , which can be simplified further by using the identity $\Psi(1/2) = -\gamma - 2 \ln 2$, where γ is the Euler constant. Finally,

$$\Lambda = -\frac{\gamma + \ln 2}{4} = -0.317590711365\dots \tag{93}$$

Our numerical results suggest that the strong-disorder expansion involves powers of $(\ln \beta)^{-1}$ rather than β^{-1} :

$$\lambda(\beta) = \frac{1}{2} \ln \beta + \Lambda + \sum_{k=1}^\infty b_k (\ln \beta)^{-k}. \tag{94}$$

Of course, it is hardly possible to probe higher-order logarithmic terms numerically. However, plotting $\lambda - \frac{1}{2} \ln \beta$ versus $(\ln \beta)^{-1}$ gives a fairly straight line for $\beta > 10^4$, with the slope $b_1 \approx 0.557$, and a perfect fit to the above functional form, keeping a quadratic term in $(\ln \beta)^{-1}$, with $b_2 \approx -0.52$ (see figure 3).

Extrapolating the quadratic fit to $(\ln \beta)^{-1} = 0$ yields $\Lambda \approx -0.318$, in good agreement with the theoretical prediction (94). In contrast to the Fibonacci case, the curve of $\lambda(\beta)$ appears perfectly smooth, and is certainly not fractal.

Finally, we note that asymptotic methods [22] should in principle allow us to perform the strong-disorder expansion more systematically.

5. Discussion

The rich behaviour exhibited by random Fibonacci numbers suggests avenues for further investigation. For instance, how do we reconcile perturbative results in the large- β limit with non-perturbative results for $\beta = 1$? This question is important as there appears to be just a single threshold $\beta = 1/4$, and therefore $\beta = 1$ lies within the large- β domain. This

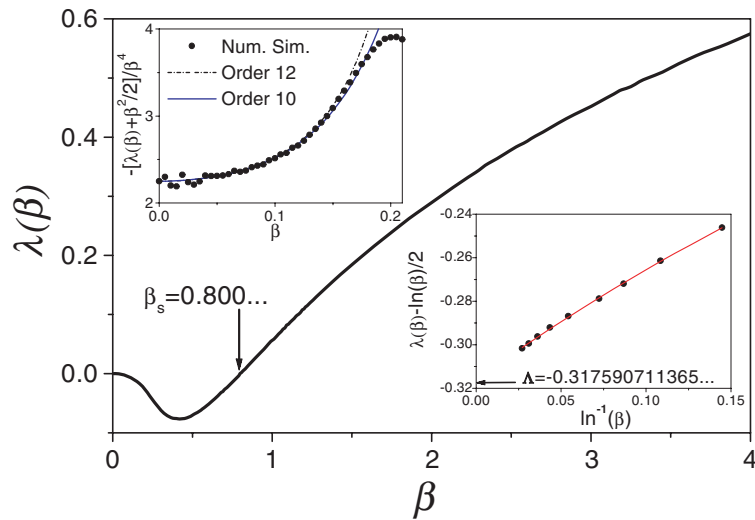


Figure 3. We plot $\lambda(\beta)$ for Gaussian random sequences. The upper inset compares $-(\lambda(\beta) + \beta^2/2)/\beta^4 = 9/4 + 22\beta^2 + \dots$ obtained from the 10th- and 12th-order expansions in equation (86) to the result of numerical simulations. The lower inset is a quadratic fit in $(\ln \beta)^{-1}$ (see equation (94)) of $\lambda(\beta) - \frac{1}{2} \ln \beta$, for large β , leading to an extrapolated $\Lambda \approx -0.318$, with $b_1 \approx 0.557$ and $b_2 \approx -0.52$.

(This figure is in colour only in the electronic version)

suggests qualitatively similar behaviours, which is not the case. The major difference between $\beta = 1$ and $\beta \rightarrow \infty$ cases is manifested in extreme behaviours of the invariant measure. In the former case, it exhibits exponential asymptotics, equations (44) and (45), while the latter is characterized by power-law asymptotics, $\nu(r) \sim r$ for $r \rightarrow 0$ and $\frac{1}{2} - \nu(r) \sim r^{-1}$ for $r \rightarrow \infty$. More generally, our perturbative results are infinitely smooth, in a gross disagreement with the behaviour for $\beta = 1$. Another (related) set of questions concerns the curve $\lambda(\beta)$: is it a fractal? Does it become genuinely smooth at least for sufficiently large β ?

One could ask for a more complete characterization of the growth (decay) of the random Fibonacci numbers. A natural conjecture is $x_n \sim e^{\lambda n} n^\omega$. For the neutrally stable recurrence, $x_{n+1} = x_n \pm \beta_s x_{n-1}$ with $\beta_s \approx 0.70258$ chosen so that $\lambda(\beta_s) = 0$, the above conjecture would imply the power-law behaviour $x_n \sim n^{\omega_s}$.

Finally, it would be very interesting to analyse random Fibonacci numbers for the critical strength of disorder, $\beta = 1/4$. This strength of disorder appears a little more interesting than $\beta = 1$: it is hard to see what distinguishes $\beta = 1$ from say $\beta = 1.23456$, while the case of $\beta = 1/4$ is certainly special as it separates the regions of weak and strong disorder.

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